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A Vector Generalisation of de Montessus' Theorem for the Case of Polar Singularities on the Boundary

D. E. Roberts

Department of Mathematics, Napier University, 10 Colinton Road, Edinburgh EH10 5DT, Scotland
E-mail: d.roberts@napier.ac.uk

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In the context of vector Padé approximants we present an extension of de Montessus' theorem to vector-valued meromorphic functions with poles on the boundary of interest—thus strengthening previous results for these approximants. The proofs are framed in Clifford algebras which provide a natural language for discussing vector rational approximants. We also present results for the asymptotic behaviour of the constituent parts of the Clifford denominator—namely, its scalar and bivector parts. In particular, for the case of vector-valued *rational* functions in which the principal parts are orthogonal to each other for different poles, we demonstrate that the rate of convergence is doubled for the scalar part of the denominator. Finally, we derive consequences of the convergence theorems for the approximation of poles, using either the complete Clifford denominator or its scalar part.

1. INTRODUCTION

In this paper we are concerned with a generalisation of Padé approximation to the case of vector-valued functions $\mathbf{f}: \mathbb{C} \mapsto \mathbb{C}^d$ analytic at the origin. We consider the power series

$$\mathbf{f}(z) = \mathbf{c}_0 + z\mathbf{c}_1 + z^2\mathbf{c}_2 + \cdots, \qquad \mathbf{c}_n \in \mathbb{C}^d, \quad z \in \mathbb{C} \tag{1.1}$$

which converges in some neighbourhood of the origin. In particular, we consider meromorphic functions of the form

$$\mathbf{f}(z) = \frac{\mathbf{g}_\kappa(z)}{R_\kappa(z)}, \tag{1.2}$$

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where $R_\kappa(z)$ is a polynomial over $\mathbb{C}$ with zeroes $z_1, \ldots, z_\kappa$, and each component $g_{\kappa,i}(z)$ $(i = 1, \ldots, d)$ is analytic for $|z| < |z_{\kappa+1}|$ with *only polar singularities* on $|z| = |z_{\kappa+1}|$. This type of function occurs in the study of iterative methods of solving systems of linear algebraic equations; see, e.g., [4, 29] and section 4 of this work. κ controls the number of poles considered in the approximation.

The rational approximations considered in this paper stem from the work of Wynn [31,32] and later investigated by Graves-Morris *et al.* [3]. However, there is more than one extension of Padé approximation to vector-valued functions cf. [10, 28]—in particular we refer to the work of Sidi. Here we investigate those approximants derived using the vector inverse of $\mathbf{v} := (v_1, v_2, \ldots, v_d) \in \mathbb{C}^d$ defined by

$$\mathbf{v}^{-1} := \frac{\mathbf{v}}{\mathbf{v} \cdot \mathbf{v}} \tag{1.3}$$

provided $\mathbf{v} \cdot \mathbf{v} := \sum_{i=1}^{d} v_i^{\,2} \neq 0$. For $d = 1$ we obtain the usual Padé approximants. There have been a number of studies of the convergence behaviour of rational approximants to vector-valued meromorphic functions, e.g., [8–10, 20, 28]. Indeed, Sidi has considered the case of meromorphic functions with poles on the circle of meromorphy both for the scalar case [26] and the vector case [25, 28] using a definition of vector-valued rational approximant different from that employed here. For $d = 1$ our results are similar to those of Sidi [26], and Liu [14], who has also studied this problem. As far as the vector case is concerned, there are analogous results with Sidi's work—e.g., Theorem 3.1 has counterparts in Eq. (4.7) of Theorem 4.1 and Eq. (4.21) of Theorem 4.2 of [28]. This, despite the fact that the structure of the approximants studied in this paper is quite different from that of the constructs in [28].

The natural framework for discussing vector-valued rational approximants based on (1.3) is that of Clifford algebras—which allows multiplication, as well as addition, of vectors and multiplication by scalars. This approach has the advantage of enabling certain arguments valid in the scalar case to be translated to the vector case. In fact, to prove our first theorem we extend Saff's method [24] as presented in [2] by Baker and Graves-Morris.

In the next section we define vector Padé approximants in the context of Clifford algebras and state some properties required later. Then, in Section 3, we present a theorem generalising de Montessus' result to vector-valued meromorphic functions. We also present consequences for the numerator polynomial. The presence of only polar singularities on the boundary $|z| = |z_{\kappa+1}|$, allows the derivation of stronger results than those published hitherto [10, 20].

In the fourth section we concentrate on the denominator of the $[l/m]$ vector Padé approximant. This polynomial is completely determined by a scalar polynomial of maximum degree m and an anti-symmetric matrix of order d (corresponding to a bivector), whose entries are also polynomials of maximum degree m [23]. We then consider the case where each component of $\mathbf{f}(z)$ is itself rational. This type of function occurs in the context of generating matrices and is discussed in detail by Sidi in [27]. We prove that the rate of convergence of the scalar polynomial to $R_\kappa(z)$ is doubled in those situations where the principal parts of the function $\mathbf{f}(z)$ corresponding to different poles are orthogonal to each other, thus demonstrating that the convergence behaviour for $d>1$ can be stronger than that for $d=1$. For the approximants considered here this has been observed numerically in certain simple cases, in [5], and explained in [21]. The current work extends analogous results of [28] from simple to multiple poles. To complete the study of the denominator in this section, we consider the convergence behaviour of the anti-symmetric matrix of polynomials as it tends to the null matrix.

In the last section, we investigate, à la Sidi [28], the asymptotic character of the errors in pole approximation using the denominators of vector Padé approximants. We not only consider the effects of using the full Clifford denominator but also of its scalar part in this approximation.

2. CLIFFORD ALGEBRAS AND VECTOR PADÉ APPROXIMANTS

Let $\{\mathbf{e}_1, \mathbf{e}_2 \cdots \mathbf{e}_d\}$ be an orthonormal basis of $\mathbb{R}^d$ satisfying

$$\mathbf{e}_i\mathbf{e}_j + \mathbf{e}_j\mathbf{e}_i = 2\delta_{i,j}, \qquad i, j = 1, 2, \ldots, d. \tag{2.1}$$

The elements $\mathbf{e}_i$, $i=1, \ldots, d$ generate the complex Clifford algebra $Cl(\mathbb{C}^d)$ with unity denoted by 1 [17, 18]. We assume that the universality property, $\mathbf{e}_1\mathbf{e}_2 \cdots \mathbf{e}_d \neq \pm 1$, is valid. Thus $Cl(\mathbb{C}^d)$ is a 2^d-dimensional linear space over $\mathbb{C}$ with basis elements

$$\mathbf{e}_I = \mathbf{e}_{i_1 i_2 \cdots i_k} = \mathbf{e}_{i_1}\mathbf{e}_{i_2} \cdots \mathbf{e}_{i_k}, \tag{2.2}$$

where $I = \{i_1, i_2, \ldots, i_k\}$ and $1 \leqslant i_1 < i_2 < \cdots i_k \leqslant d$ for $k = 1, 2, \ldots, d$. The empty set, $I = \emptyset$, i.e., $k=0$, corresponds to the identity element. A general element of $Cl(\mathbb{C}^d)$ is given by

$$u = \sum_I a_I \mathbf{e}_I, \qquad a_I \in \mathbb{C}, \tag{2.3}$$

where the summation is over the 2^d different ordered multi-indices I.

For given k, the $\mathbf{e}_I$ form the basis of a subspace, $\bigwedge^k \mathbb{C}^d$, whose elements are called k-vectors. $Cl(\mathbb{C}^d)$ is the direct sum of the spaces $\bigwedge^k \mathbb{C}^d$ for $k = 0, 1, ..., d$. The k-vector part of the Clifford element a is denoted by $\langle a \rangle_k$. The coefficient $a_0 := \langle a \rangle_0$ is called the real or scalar part of a, and is also denoted by $Re(a)$. $\mathbb{C}^d$ may be identified with the subspace $\bigwedge^1 \mathbb{C}^d$ of $Cl(\mathbb{C}^d)$. To any vector $(v_1, v_2, ..., v_d) \in \mathbb{C}^d$ we associate the Clifford element, $\sum_{i=1}^d v_i \mathbf{e}_i$, and denote each by $\mathbf{v}$. Using (2.1) we obtain

$$\mathbf{v}^2 = \mathbf{v} \cdot \mathbf{v} \tag{2.4}$$

thus implying

$$\mathbf{v}^{-1} = \frac{\mathbf{v}}{\mathbf{v} \cdot \mathbf{v}} \tag{2.5}$$

provided $\mathbf{v} \cdot \mathbf{v} \neq 0$, cf. (1.3).

We may write $Cl(\mathbb{C}^d) = Cl^+(\mathbb{C}^d) \oplus Cl^-(\mathbb{C}^d)$ where $Cl^+(\mathbb{C}^d)$ is the even subalgebra composed of the direct sum of the spaces $\bigwedge^k \mathbb{C}^d$ for even k, while $Cl^-(\mathbb{C}^d)$ is direct sum of the spaces $\bigwedge^k \mathbb{C}^d$ for odd k.

We make use of the reverse anti-automorphism : $a \mapsto \tilde{a}$ obtained by reversing the order of factors in $\mathbf{e}_I$; hence, $\widetilde{ab} = \tilde{b}\tilde{a}$. We note that $\tilde{\mathbf{v}} = \mathbf{v}$, for $\mathbf{v} \in \mathbb{C}^d$.

The spinor norm or absolute value of an element in $Cl(\mathbb{C}^d)$ is a generalisation of the Euclidean norm on $\mathbb{C}^d$

$$|a| = \sqrt{\sum_I |a_I|^2}, \qquad a \in Cl(\mathbb{C}^d). \tag{2.6}$$

From this definition it is clear that $|\tilde{a}| = |a|, \forall a \in Cl(\mathbb{C}^d)$. From [11] we have

$$|uv| \leqslant K_d \, |u| \, |v|, \qquad u, v \in Cl(\mathbb{C}^d), \tag{2.7}$$

where

$$K_d = 2^{d/4} \quad \text{for } d \text{ even} \quad \text{and} \quad K_d = 2^{(d+1)/4} \quad \text{for } d \text{ odd.} \tag{2.8}$$

Since $Cl(\mathbb{C}^d)$ is a finite-dimensional normed linear space it is complete.

Our main interest in this paper is in functions $\mathbf{f}\colon \mathbb{C} \mapsto \mathbb{C}^d$, but we will find that we have to extend our considerations to functions $f\colon \mathbb{C} \mapsto Cl(\mathbb{C}^d)$. All linear operators such as differentiation and integration may be implemented componentwise. Hence, a function, $f(z) := \sum_I f_I(z)\, \mathbf{e}_I$, is analytic in a domain D if each component function $f_I(z)$ is analytic in D. Since the algebra is finite-dimensional it follows that for functions $g(z), h(z)$: $\mathbb{C} \mapsto Cl(\mathbb{C}^d)$, both analytic in D, their product $f(z) := g(z)\, h(z)$ is also analytic in D. Here, we have $f_I(z) = \sum_{A,B} g_A(z)\, h_B(z)$, the summation being over A, B such that $\mathbf{e}_A \mathbf{e}_B = \mathbf{e}_I$. A function $f(z)$ analytic in $\{z \in \mathbb{C} : |z| < \rho\}$ may be expanded in a MacLaurin series

$$f(z) = \sum_{n=0}^{\infty} c_n z^n, \tag{2.9}$$

where $c_n := f^{(n)}(0)/n!$, $n = 0, 1, \ldots$—the infinite series being convergent in the spinor norm for $|z| < \rho$. Hermite's formula for a partial sum of this series is given by

$$\frac{1}{2\pi i} \int_{\Gamma} \frac{1-(z/v)^{l+1}}{v-z} f(v)\, dv, \qquad \Gamma \in D, \tag{2.10}$$

where D is a domain of analyticity of $f(z)$ and Γ is a contour enclosing the origin; this is valid since it is true componentwise.

The right-handed $[l/m]$ vector Padé approximant [19] to the vector-valued function $\mathbf{f}(z)$, which is a particular case of (2.9), is given by

$$[l/m](z) := p^{[l/m]}(z)\{q^{[l/m]}(z)\}^{-1}, \tag{2.11}$$

where $p^{[l/m]}(z)$ and $q^{[l/m]}(z)$ are polynomials over $Cl(\mathbb{C}^d)$ of maximum degrees l, m respectively, such that

$$[\mathbf{f}(z) - [l/m](z)]_0^{l+m} = 0, \tag{2.12}$$

where we have used Nuttall's notation for the MacLaurin section of $\phi(z) := \sum_{i=0}^{\infty} \phi_i z^i$

$$[\phi(z)]_0^n := \sum_{i=0}^{n} \phi_i z^i. \tag{2.13}$$

The leading coefficient of a polynomial is denoted by a *dot*, while a *hat* is used to indicate that a polynomial is monic. Hence, we have

$$\dot{\hat{q}}^{[l/m]}(z) = 1. \tag{2.14}$$

Because the *complex* Clifford algebra $Cl(\mathbb{C}^d)$ is used, the left-handed vector Padé approximant is given by the reversion of the right-handed form. The $[l/m]$ vector Padé approximant, if it exists, is unique. Furthermore, we may define vector and scalar polynomials (of maximum degrees $(l+m)$ and $2m$, respectively) as

$$\mathbf{P}^{l,m}(z) := p^{[l/m]}(z)\, \widetilde{q^{[l/m]}(z)} \in \mathbb{C}^d[z] \tag{2.15}$$

and

$$Q^{l,m}(z) := q^{[l/m]}(z)\, \widetilde{q^{[l/m]}(z)} \in \mathbb{C}[z]. \tag{2.16}$$

The reader is referred to [6, 19, 22] for justification and further discussion of the properties of vector Padé approximants.

It may be shown that, with the normalisation (2.14), $q^{[l/m]}(z) \in Cl^{+}(\mathbb{C}^d)[z]$ and $p^{[l/m]}(z) \in Cl^{-}(\mathbb{C}^d)[z]$. Furthermore, using a theorem of Lipschitz dating from 1886 [12, 13], it is possible to demonstrate that the denominator polynomial may be determined by its scalar and bivector parts only [23]:

$$\sigma^{[l/m]}(z) := \langle q^{[l/m]}(z) \rangle_0 \tag{2.17}$$

and

$$\Lambda^{[l/m]}(z) := \langle q^{[l/m]}(z) \rangle_2. \tag{2.18}$$

In Section 4 we consider the asymptotic behaviour of these quantities as $l \to \infty$.

There is a corresponding characterisation for the numerator polynomial [23].

3. VECTOR-VALUED MEROMORPHIC FUNCTIONS

We consider vector-valued meromorphic functions of the form

$$\mathbf{f}(z) := \frac{\mathbf{g}(z)}{R_M(z)}, \tag{3.1}$$

where

$$R_k(z) := \prod_{i=1}^{k} (z - z_i)^{\mu_i} \quad \text{for} \quad k = 1, 2, ..., M \tag{3.2}$$

in which the μ_i are positive integers and the z_i are distinct complex numbers ordered as follows

$$\begin{aligned} 0 < |z_1| \leqslant |z_2| \leqslant \cdots \leqslant |z_\kappa| < \rho_\kappa < |z_{\kappa+1}| = |z_{\kappa+2}| \\ = \cdots = |z_{\kappa+h}| < \rho_{\kappa+h} < |z_{\kappa+h+1}| \leqslant \cdots \leqslant |z_M| < \rho. \end{aligned} \tag{3.3}$$

The ρ_κ, $\rho_{\kappa+h}$ are positive numbers satisfying (3.3). Each component function $g_i(z)$, $i = 1, \ldots, d$ is analytic for $|z| < \rho$, with the vector function satisfying

$$\mathbf{g}(z_i) \cdot \mathbf{g}(z_i) \neq 0, \qquad i = 1, 2, \ldots, M \tag{3.4}$$

thus guaranteeing the invertibility of each complex vector $\mathbf{g}(z_i)$—cf. (2.5).

From its definition, $\mathbf{f}(z)$ has a MacLaurin series expansion of the form (2.1) valid for $|z| < |z_1|$. Our interest is in approximating $\mathbf{f}(z)$ as far as the pole z_κ, for which $|z_\kappa| < |z_M|$. We set

$$m := \sum_{i=1}^{\kappa} \mu_i \qquad \text{and} \qquad \mathbf{g}_\kappa(z) := R_\kappa(z)\, \mathbf{f}(z) \tag{3.5}$$

with

$$\bar{\mu}_k := \max_{|z_i| = |z_k|} \mu_i, \qquad k = 1, \ldots, M. \tag{3.6}$$

Given a subset A of $\mathbb{C}$ and a Clifford-valued function $a\colon \mathbb{C} \mapsto Cl(\mathbb{C}^d)$, we adopt the notation

$$\|a(z)\|_A := \max_{z \in A} |a(z)|$$

using the spinor norm (2.6). Finally, we define

$$D_k := \{z \in \mathbb{C} : |z| < |z_k|\}, \qquad k = 1, \ldots, M. \tag{3.7}$$

THEOREM 3.1. *For a vector-valued function,* $\mathbf{f}(z)$, *of the form described above, the* $[l/m]$ *vector Padé approximant,* $p^{[l/m]}(z)\{q^{[l/m]}(z)\}^{-1}$, *exists for sufficiently large* $n := l + m$. *Furthermore, as* $n \to \infty$,

(i) *the monic denominators,* $\hat{q}^{[l/m]}(z)$, *converge uniformly to* $R_\kappa(z)$ *in compact subsets E of the complex plane. The rate of convergence is governed by*

$$\|R_\kappa(z) - \hat{q}^{[l/m]}(z)\|_E = O\left(n^{\bar{\mu}_\kappa + \bar{\mu}_{\kappa+1} - 2} \left|\frac{z_\kappa}{z_{\kappa+1}}\right|^n\right); \tag{3.8}$$

(ii) *the* $[l/m]$ *vector Padé approximants converge uniformly to* $\mathbf{f}(z)$ *in compact subsets S of* $D_{\kappa+1}^- := D_{\kappa+1} - \bigcup_{i=1}^{\kappa} \{z_i\}$. *In fact, there exists a real number* K_S *and an integer* n_S *such that*

$$|\mathbf{f}(z) - [l/m](z)| \leqslant K_S n^{\bar{\mu}_{\kappa+1} - 1} \left|\frac{z}{z_{\kappa+1}}\right|^n \qquad \forall z \in S, \quad \forall n > n_S. \tag{3.9}$$

Proof. We follow the approach of Saff [24] as formulated in [2] and begin by parametrising $q^{[l/m]}(z)$ as

$$q^{[l/m]}(z) = q_{n,m} R_\kappa(z) + \sum_{k=1}^{\kappa} \sum_{s=0}^{\mu_k - 1} q_n^{(s)}(z_k)\, B_{k,s}(z), \tag{3.10}$$

where $q_{n,m} \in \mathbb{R}$, $q_n^{(s)}(z_k) \in Cl(\mathbb{C}^d)$ and

$$B := \{B_{k,s}(z) : k = 1, 2, \ldots, \kappa;\ s = 0, 1 \cdots \mu_k - 1\} \tag{3.11}$$

is a Hermite basis of polynomials over $\mathbb{C}$, each of maximum degree $(m-1)$, satisfying

$$\left[\frac{d^j}{dz^j} B_{k,s}(z)\right]_{z=z_i} = \delta_{ik}\delta_{js}, \qquad 1 \leqslant i \leqslant \kappa, \quad 0 \leqslant j \leqslant \mu_i - 1. \tag{3.12}$$

From

$$q^{[l/m]}(z) = \sum_I q_I^{[l/m]}(z)\, \mathbf{e}_I$$

we obtain the scalar relations

$$q_I^{[l/m]}(z) = \sum_{k=1}^{\kappa} \sum_{s=0}^{\mu_k - 1} q_{n,I}^{(s)}(z_k)\, B_{k,s}(z), \qquad I \neq \varnothing$$

$$q_0^{[l/m]}(z) = q_{n,m} R_\kappa(z) + \sum_{k=1}^{\kappa} \sum_{s=0}^{\mu_k - 1} q_{n,0}^{(s)}(z_k)\, B_{k,s}(z), \qquad I = \varnothing.$$

We seek a solution to the approximant problem for which the denominator polynomial has a real, positive leading coefficient, $q_{n,m}$, which satisfies the normalisation

$$q_{n,m} + \sum_{k=1}^{\kappa} \sum_{s=0}^{\mu_k - 1} |q_n^{(s)}(z_k)| = 1, \tag{3.13}$$

where the spinor norm is used.

Since each component of the Clifford product $\mathbf{g}_\kappa(z)\, q^{[l/m]}(z)$ is analytic in $D_{\kappa+1}$, the first $n+1$ terms of the MacLaurin series of this product define a polynomial $\pi_n : \mathbb{C} \mapsto Cl(\mathbb{C}^d)$, which, using Hermite's formula (2.10), may be expressed as

$$\pi_n(z) = \frac{1}{2\pi i} \oint_{|v| = \rho_\kappa} \frac{1 - (z/v)^{n+1}}{v - z}\, \mathbf{g}_\kappa(v)\, q^{[l/m]}(v)\, dv. \tag{3.14}$$

From (2.12) and (3.1) we may establish the identity

$$\pi_n(z) = R_\kappa(z)\, p^{[l/m]}(z). \tag{3.15}$$

Hence,

$$\pi_n^{(s)}(z_k) = 0, \qquad k = 1, 2, ..., \kappa, \qquad s = 0, 1, ..., \mu_k - 1 \tag{3.16}$$

the superscript in which denotes the sth derivative. Since $Cl(\mathbb{C}^d)$ is assumed to be a universal algebra the $\mathbf{e}_I$ in (2.3) are linearly independent. Hence, each component $\pi_{n,I}^{(s)}(z_k)$ vanishes.

We demonstrate that Eqs. (3.16) together with the normalisation (3.13) determine all the parameters of $q^{[l/m]}(z)$ for sufficiently large l.

When imposing the conditions (3.16) we make use of the result

$$\left[\frac{d^s}{dz^s}\left\{\frac{z^{n+1}}{2\pi i}\oint_{|v|=\rho_\kappa}\frac{\mathbf{g}_\kappa(v)\, q^{[l/m]}(v)}{v^{n+1}(v-z)}\, dv\right\}\right]_{z=z_k} = O\left(n^{s+\bar{\mu}_{\kappa+1}-1}\left|\frac{z_k}{z_{\kappa+1}}\right|^n\right) \tag{3.17}$$

for $k = 1, 2, ..., \kappa$ and $s = 0, 1, ...,$ which we proceed to prove.

If $h^{l,m}(v) := \mathbf{g}_\kappa(v)\, q^{[l/m]}(v)$ then each complex-valued function $h_I^{l,m}(v)\, v^{-n-1}$ is continuous on $|v| = \rho_\kappa$, thus allowing componentwise differentiation under the integral sign [1]. Then, using Leibnitz's theorem the left-hand side of (3.17) may be shown to equal

$$s! \sum_{r=0}^{s} \binom{n+1}{s-r} z_k^{\,r-s} F_{n,r}(z_k), \tag{3.18}$$

where

$$F_{n,r}(z) := \frac{z^{n+1}}{2\pi i}\oint_{|v|=\rho_\kappa}\frac{\mathbf{g}_\kappa(v)\, q^{[l/m]}(v)}{v^{n+1}(v-z)^{r+1}}\, dv, \qquad r = 0, 1, ..., \mu_\kappa - 1, \quad |z| < \rho_\kappa. \tag{3.19}$$

Expanding the contour to $|v| = \rho_{\kappa+h}$ we may apply Cauchy's theorem to each component function to obtain

$$F_{n,r}(z) := \frac{z^{n+1}}{2\pi i}\oint_{|v|=\rho_{\kappa+h}}\frac{\mathbf{g}_\kappa(v)\, q^{[l/m]}(v)}{v^{n+1}(v-z)^{r+1}}\, dv - z^{n+1} \times \sum_{i=1}^{h} \operatorname{Res}\left[\frac{h^{l,m}(v)}{v^{n+1}(v-z)^{r+1}} : v = z_{\kappa+i}\right] \tag{3.20}$$

since the singularities of $h_I^{l,m}(v)$ encountered are those of $\mathbf{g}_\kappa(v)$ which are poles at $z_{\kappa+i}$ each of multiplicity $\mu_{\kappa+i}$, $(i = 1, 2, ..., h)$—cf. (3.2).

Considering the integral first, we note that the polynomials $B_{\kappa,s}(z)$, $R_\kappa(z)$ in (3.10) are bounded, by $B_{\kappa+h}$ say, independently of l, on $|v| = \rho_{\kappa+h}$. It then follows from (3.10) that $\max_{|v|=\rho_{\kappa+h}} |q^{[l/m]}(v)| \leqslant B_{\kappa+h}$. Furthermore, since each component $g_{\kappa,i}(v)$, $i = 1, 2, \ldots, d$ is continuous and therefore bounded on $|v| = \rho_{\kappa+h}$, $|\mathbf{g}_\kappa(v)|$ is also bounded on this circle, by G_κ say. Hence, using (2.7)

$$|h^{l,m}(v)| \leqslant K_d G_\kappa B_{\kappa+h}, \qquad |v| = \rho_{\kappa+h}.$$

Therefore, the spinor norm of the first term on the right-hand side of (3.20) is bounded by

$$C_\kappa \left| \frac{z}{\rho_{\kappa+h}} \right|^n, \qquad |z| < \rho_\kappa, \tag{3.21a}$$

where

$$C_\kappa := \max_{0 \leqslant r \leqslant \mu_\kappa - 1} \frac{\rho_\kappa K_d G_\kappa B_{\kappa+h}}{(\rho_{\kappa+h} - \rho_\kappa)^{r+1}} \tag{3.21b}$$

a constant independent of l.

Considering the ith term in the summation of (3.20) (including the factor z^{n+1}), suppose $h_I^{l,m}(v)$ has a pole at $v = z_{\kappa+i}$ of order $s_i + 1$. The corresponding residue is given by

$$\frac{z^{n+1}}{s_i!} \lim_{v \to z_{\kappa+i}} \frac{d^{s_i}}{dv^{s_i}} \left\{ \frac{(v - z_{\kappa+i})^{s_i+1} h_I^{l,m}(v)}{v^{n+1}(v-z)^{r+1}} \right\}. \tag{3.22}$$

For fixed z with $|z| < \rho_\kappa$, each component $g_{\kappa,j}(v)(v - z_{\kappa+i})^{s_i+1}(v-z)^{-r-1}$, $j = 1, 2, \ldots, d$ is an analytic function of v at $v = z_{\kappa+i}$. So, this function and its first s_i derivatives are bounded at this point. Note also that the polynomials $q_J^{[l/m]}(v)$ (for all multi-indices J), of maximum degree m, and their derivatives may be bounded at the same point, using (3.10), (3.13), by a constant independent of l. Hence, with the help of Leibnitz's theorem and equation (2.7), we may find constants C_i, independent of n and z such that (3.22) is bounded by

$$C_i n^{s_i} \left| \frac{z}{z_{\kappa+1}} \right|^n \quad \text{for} \quad |z| < \rho_\kappa \quad \text{and} \quad i = 1, 2, \ldots, h \tag{3.23}$$

for sufficiently large n. Now, there must exist a multi-index I such that $h_I^{l,m}(v)$ has a pole at $v = z_{\kappa+i}$ of exact order $\mu_{\kappa+i}$. If this were not the case, the invertibility of $q^{[l/m]}(z_{\kappa+i})$ would still follow from the argument below, since only the power of n is affected. This would imply that

$\mathbf{g}_\kappa(v)(v-z_{\kappa+i})^{\mu_{\kappa+i}}$ vanishes at $v=z_{\kappa+i}$, contradicting the assumption (3.4). Consequently, there is a constant C, also independent of n and z such that

$$|F_{n,r}(z)| \leqslant Cn^{\bar{\mu}_{\kappa+1}-1}\left|\frac{z}{z_{\kappa+1}}\right|^n, \qquad |z|<\rho_\kappa \tag{3.24}$$

for sufficiently large n.

Noting that $s!\,\binom{n+1}{s-r}=O(n^{s-r})$ we obtain the result (3.17).

We are now in a position to estimate the parameters describing $q^{[l/m]}(z)$ and to demonstrate that (3.13) may be satisfied. On imposing the constraints (3.16) we may use (3.17) to write

$$\mathbf{g}_\kappa(z_k)\,q_n^{(s)}(z_k)+\sum_{r=0}^{s-1}\binom{s}{r}\mathbf{g}_\kappa^{(s-r)}(z_k)\,q_n^{(r)}(z_k)=O\left(n^{s+\bar{\mu}_{\kappa+1}-1}\left|\frac{z_k}{z_{\kappa+1}}\right|^n\right) \tag{3.25}$$

for $k=1, 2, ..., \kappa$ and $s=0, 1, ..., \mu_k-1$.

Since $\mathbf{g}_\kappa(z_k)$ is invertible, from (3.4), we deduce that, with $s=0$,

$$q_n^{(0)}(z_k)=O\left(n^{\bar{\mu}_{\kappa+1}-1}\left|\frac{z_k}{z_{\kappa+1}}\right|^n\right). \tag{3.26}$$

Then, by induction, we may demonstrate that

$$q_n^{(s)}(z_k)=O\left(n^{s+\bar{\mu}_{\kappa+1}-1}\left|\frac{z_k}{z_{\kappa+1}}\right|^n\right) \tag{3.27}$$

for $k=1, 2, ..., \kappa$ and $s=0, 1, ..., \mu_k-1$ (for $s\geqslant\mu_k$ see Eq. (3.41)).

From (3.27), (3.13), and (3.3) we conclude that

$$q_{n,m}=1-\sum_{k=1}^{\kappa}\sum_{s=0}^{\mu_k-1}|q_n^{(s)}(z_k)|=1+O\left(n^{\bar{\mu}_\kappa+\bar{\mu}_{\kappa+1}-2}\left|\frac{z_\kappa}{z_{\kappa+1}}\right|^n\right) \tag{3.28}$$

so that, as required, for sufficiently large n, the leading coefficient of $q^{[l/m]}(z)$ is positive.

We now consider the monic denominator polynomial $\hat{q}^{[l/m]}(z)$

$$\hat{q}^{[l/m]}(z):=\frac{q^{[l/m]}(z)}{q_{n,m}}$$

for $z\in E$, a compact subset of the complex plane. Clearly, the polynomials $R_\kappa(z)$, $B_{k,s}(z)$, $(k=1, 2, ..., \kappa;\ s=0, 1, ..., \mu_k-1)$ are bounded on E. This, together with (3.27) and (3.28) implies the existence of a constant K_E, independent of n and z such that, for sufficiently large n,

$$|R_\kappa(z) - \hat{q}^{[l/m]}(z)| = \frac{1}{q_{n,m}} \left| \sum_{k=1}^{\kappa} \sum_{s=0}^{\mu_k - 1} q_n^{(s)}(z_k)\, B_{k,s}(z_k) \right| \leqslant K_E n^{\bar{\mu}_\kappa + \bar{\mu}_{\kappa+1} - 2} \left| \frac{z_\kappa}{z_{\kappa+1}} \right|^n, \quad z \in E \tag{3.29}$$

thus guaranteeing the uniform convergence of $\hat{q}^{[l/m]}(z)$ to $R_\kappa(z)$ in compact subsets, E, of $\mathbb{C}$ at the rate stated in (3.8).

We now prove that, for n large enough, the monic denominator polynomial is invertible in compact subsets of $D_{\kappa+1}^-$. Given a compact subset S of $D_{\kappa+1}^-$ there is a positive number ϵ such that $|R_\kappa(z)| > \epsilon$ for all z in S. Furthermore, given $0 < \delta < 1$, there exists an integer l_0, such that

$$|R_\kappa(z) - \hat{q}^{[l/m]}(z)| < \frac{\epsilon\delta}{K_d} \qquad \forall l > l_0 \qquad \text{and} \qquad z \in S$$

which leads to

$$\left| 1 - \frac{\hat{q}^{[l/m]}(z)}{R_\kappa(z)} \right| < \frac{\delta}{K_d} \qquad \forall l > l_0 \qquad \text{and} \qquad z \in S.$$

Now, it is readily shown, using standard methods, e.g., [30], that, for $a \in Cl(\mathbb{C}^d)$ satisfying $|a| < \delta/K_d$ where $\delta < 1$, then $(1-a)^{-1}$ exists in $Cl(\mathbb{C}^d)$ and $|(1-a)^{-1}| < 1/(1-\delta)$. The proof uses the completeness of $Cl(\mathbb{C}^d)$ and the result $|a^{r+1}| \leqslant K_d{}^r\, |a|^{r+1}$ which follows from (2.7). Identifying $(1-a)$ with $\hat{q}^{[l/m]}(z)/R_\kappa(z)$ we conclude that $\{q^{[l/m]}(z)\}^{-1}$ exists, for $l > l_0$, $z \in S$, and that

$$|\{q^{[l/m]}(z)\}^{-1}| < \frac{1}{\epsilon(1-\delta)} \qquad \forall z \in S \quad \forall l > l_0. \tag{3.30}$$

We define the polynomial over $Cl(\mathbb{C}^d)$ of degree l

$$p^{[l/m]}(z) := \frac{\pi_n(z)}{R_\kappa(z)\, q_{n,m}} \tag{3.31}$$

and consider the rational function

$$p^{[l/m]}(z)\{\hat{q}^{[l/m]}(z)\}^{-1} = \frac{\pi_n(z)}{R_\kappa(z)} \{q^{[l/m]}(z)\}^{-1} \tag{3.32}$$

which is well-defined for $l > l_0$ and any $z \in S$. We observe that, from (3.14), (3.32),

$$\mathbf{f}(z) - p^{[l/m]}(z)\{\hat{q}^{[l/m]}(z)\}^{-1} = \frac{z^{n+1}}{2\pi i R_\kappa(z)} \oint_{|v|=\rho_\kappa} \frac{\mathbf{g}_\kappa(v)\, q^{[l/m]}(v)}{v^{n+1}(v-z)}\, dv \{q^{[l/m]}(z)\}^{-1} \tag{3.33}$$

thus satisfying the Padé order condition (2.12). Hence, the $[l/m]$ vector Padé approximant to $\mathbf{f}(z)$ is given by (3.32). Since the right-hand side of (3.33) is identical to

$$\frac{F_{n,0}(z)}{R_\kappa(z)} \{q^{[l/m]}(z)\}^{-1}$$

we may use (3.24), (3.30) to bound the vector Padé error

$$|\mathbf{f}(z) - [l/m](z)| \leqslant K_S n^{\bar{\mu}_{\kappa+1}-1} \left| \frac{z}{z_{\kappa+1}} \right|^n \qquad \forall z \in S \qquad \forall l > l_0, \tag{3.34}$$

where

$$K_S := \frac{C}{\epsilon^2(1-\delta)} \tag{3.35}$$

independent of n and z; thus, the result (3.9) is established. ∎

COROLLARY 3.2. *As $n \to \infty$, the numerators, $p^{[l/m]}(z)$, converge uniformly to $\mathbf{g}_\kappa(z)$ in compact subsets T of $D_{\kappa+1}$.*

In fact, if $|z_\kappa| < \sigma_1 \leqslant |z| \leqslant \sigma_2 < |z_{\kappa+1}|$, then there is a constant $K_{1,2}$ and an integer $n_{1,2}$ such that

$$|\mathbf{g}_\kappa(z) - p^{[l/m]}(z)| \leqslant K_{1,2} n^{\bar{\mu}_{\kappa+1}-1} \left| \frac{z}{z_{\kappa+1}} \right|^n \qquad \forall \sigma_1 \leqslant |z| \leqslant \sigma_2, \quad \forall n > n_{1,2}. \tag{3.36}$$

Whereas, if $|z| \leqslant \sigma_3 < |z_\kappa|$, then there are numbers K_3, n_3 such that

$$|\mathbf{g}_\kappa(z) - p^{[l/m]}(z)| \leqslant K_3 n^{\bar{\mu}_\kappa + \bar{\mu}_{\kappa+1}-2} \left| \frac{z_\kappa}{z_{\kappa+1}} \right|^n \qquad \forall |z| \leqslant \sigma_3, \quad n > n_3. \tag{3.37}$$

Proof. Define $S_{1,2} := \{z \in \mathbb{C} : \sigma_1 \leqslant |z| \leqslant \sigma_2\}$ where $|z_\kappa| < \sigma_1 < \sigma_2 < |z_{\kappa+1}|$. Then $S_{1,2} \subset D^-_{\kappa+1}$. Hence, $R_\kappa(z)$ is bounded above and below for $z \in S_{1,2}$, while for sufficiently large n, $q^{[l/m]}(z)$ is bounded above. We write

$$|\mathbf{g}_\kappa(z) - p^{[l/m]}(z)| \leqslant K_d \left\{ |\mathbf{f}(z) - [l/m](z)| \, |\hat{q}^{[l/m]}(z)| + \frac{|\mathbf{g}_\kappa(z)|}{|R_\kappa(z)|} |R_\kappa(z) - \hat{q}^{[l/m]}(z)| \right\}. \quad (3.38)$$

Since each component of $\mathbf{g}_\kappa(z)$ is bounded for $z \in S_{1,2}$ we may use (3.34), (3.29) to yield (3.36).

Noting that each Clifford component of the numerator error is a function analytic in $D_{\kappa+1}$, we may use the maximum modulus principle to deduce uniformity of convergence in any compact subset of $D_{\kappa+1}$.

Suppose $|z| < |z_\kappa|$, then there is a number $\sigma_3 \in (|z_{\kappa-1}|, |z_\kappa|)$ such that $|z| \leqslant \sigma_3$. Appealing to the maximum modulus principle and (3.38) with $|z| = \sigma_3$ we derive (3.37). ∎

The behaviour of the numerator error at the interpolation points is given by (3.37) for $|z_k| < |z_\kappa|$. If, however, $|z_k| = |z_\kappa|$, then we may proceed as follows. Using (3.14), (3.19) we write

$$\{\mathbf{g}_\kappa(z) - p^{[l/m]}(z)\} \, R_\kappa(z) = \mathbf{g}_\kappa(z)\{R_\kappa(z) - \hat{q}^{[l/m]}(z)\} + F_{n,0}(z)(q_{n,m})^{-1}. \quad (3.39)$$

Differentiating μ_k times with respect to z, and noting that $R_\kappa^{(\mu_k)}(z_k) \neq 0$, we obtain

$$|\mathbf{g}_\kappa(z_k) - p^{[l/m]}(z_k)| = O\left(n^{\alpha_k} \left| \frac{z_\kappa}{z_{\kappa+1}} \right|^n \right), \qquad \alpha_k := \bar{\mu}_{\kappa+1} - 1 + \max(\bar{\mu}_\kappa - 1, \mu_k) \quad (3.40)$$

using (3.2), (3.17) together with the fact that (for $m > 1$)

$$\left| \frac{d^s}{dz^s} \hat{q}^{[l/m]}(z_k) - R_\kappa^{(s)}(z_k) \right| = O\left(n^{\bar{\mu}_\kappa + \bar{\mu}_{\kappa+1} - 2} \left| \frac{z_\kappa}{z_{\kappa+1}} \right|^n \right)$$
$$\text{for} \quad s = \mu_k, \mu_k + 1, \ldots, m-1 \quad (3.41)$$

which follows from (3.10), (3.27).

From the above result and theorem (3.1) we readily prove

COROLLARY 3.3. *Let the scalar polynomial $Q^{l,m}(z)$ and the vector of polynomials $\mathbf{P}^{l,m}(z)$ be defined by* (2.15), (2.16) *with the normalisation* (2.14).

(i) *As* $n \to \infty$ *the polynomials* $Q^{l,m}(z)$ *converge uniformly to* $\{R_\kappa(z)\}^2$ *in compact subsets, E, of the complex plane. The rate of convergence is governed by*

$$\|\{R_\kappa(z)\}^2 - Q^{l,m}(z)\|_E = O\left(n^{\bar{\mu}_\kappa + \bar{\mu}_{\kappa+1} - 2}\left|\frac{z_\kappa}{z_{\kappa+1}}\right|^n\right). \tag{3.42}$$

(ii) *As* $n \to \infty$ *the vectors of polynomials* $\mathbf{P}^{l,m}(z)$ *converge uniformly to* $R_\kappa(z)\,\mathbf{g}_\kappa(z)$ *in compact subsets, T, of* $D_{\kappa+1}$.

If $|z_\kappa| < \sigma_1 \leqslant |z| \leqslant \sigma_2 < |z_{\kappa+1}|$, *then there is a constant* $K'_{1,2}$ *and an integer* $n'_{1,2}$ *such that*

$$|R_\kappa(z)\,\mathbf{g}_\kappa(z) - \mathbf{P}^{l,m}(z)| \leqslant K'_{1,2} n^{\bar{\mu}_{\kappa+1} - 1}\left|\frac{z}{z_{\kappa+1}}\right|^n \quad \forall \sigma_1 \leqslant |z| \leqslant \sigma_2, \quad n > n'_{1,2}. \tag{3.43}$$

If $|z| \leqslant \sigma_3 < |z_\kappa|$, *then there are numbers* K'_3, n'_3 *such that*

$$|R_\kappa(z)\mathbf{g}_\kappa(z) - \mathbf{P}^{l,m}(z)| \leqslant K'_3 n^{\bar{\mu}_\kappa + \bar{\mu}_{\kappa+1} - 2}\left|\frac{z_\kappa}{z_{\kappa+1}}\right|^n \quad \forall |z| \leqslant \sigma_3, \quad n > n'_3. \tag{3.44}$$

It is straightforward to extend these ideas to estimate $\{\mathbf{g}_\kappa^{(s)}(z_k): s = 0, ..., \mu_k - 1\}$ from the derivatives of $\mathbf{P}^{l,m}(z)$ at z_k from the μ_k^{th} to the $(\mu_k + s)^{th}$.

4. SCALAR AND BIVECTOR PARTS OF THE DENOMINATOR

The numerator and denominator polynomials of a vector Padé approximant appear to need many degrees of freedom in their description, cf. (2.3). However, in [23] it is shown how each may be expressed in terms of a scalar polynomial and an antisymmetric matrix of dimension d with polynomial entries. This matrix corresponds to a bivector in the Clifford algebra $Cl(\mathbb{C}^d)$. In this section we present an analysis of the asymptotic behaviour of these quantities for the denominator polynomial in the case where each $g_i(z)$, $i = 1, ..., d$ of (3.1) is a polynomial of maximum degree $\sum_{k=1}^{M+1} \mu_k$. We may then write

$$\mathbf{f}(z) = \sum_{k=1}^{M} \sum_{s=0}^{\mu_k - 1} \frac{\mathbf{v}_{k,s}}{(z - z_k)^{s+1}} + \mathbf{G}(z), \tag{4.1}$$

where each component $G_i(z)$, $i = 1, ..., d$ is a polynomial of maximum degree μ_{M+1}. This type of vector-valued function is of relevance in the study of matrix iteration methods when the matrix may be defective. The reader is referred to [27, 29] for a discussion of this topic together with a derivation of the above generating function. We are particularly interested in the behaviour of the denominator polynomial of the $[l/m]$ vector Padé approximant to $\mathbf{f}(z)$ when the vector residues $\mathbf{v}_{k,s}$ are orthogonal for different poles, i.e.,

$$\mathbf{v}_{k_1, s_1} \cdot \mathbf{v}_{k_2, s_2} = 0, \qquad k_1 \neq k_2 \tag{4.2}$$

for $s_1 = 0, 1, \ldots, \mu_{k_1} - 1$, $s_2 = 0, 1, \ldots, \mu_{k_2} - 1$.

Since (4.1) is an instance of (3.1) it is clear from theorem (3.1) that the $[l/m]$ approximant exists for sufficiently large $n := l + m$ and that the monic denominator $\hat{q}^{[l/m]}(z)$ converges uniformly to $R_\kappa(z)$ in compact subsets of $\mathbb{C}$.

Employing the notation of definitions (2.17) and (2.18), where the denominator polynomial is monic, we prove the following result:

THEOREM 4.1. *Given a vector-valued function* $\mathbf{f}(z)$ *of the form* (4.1) *with* (3.4) *and* (4.2) *valid, together with a compact subset* E *of* $\mathbb{C}$ *, then, for sufficiently large* n*, the* $[l/m]$ *vector Padé approximant exists. The monic denominator of this approximant has a scalar part which converges uniformly to* $R_\kappa(z)$ *in* E*, with the rate of convergence governed by*

$$\|R_\kappa(z) - \sigma^{l,m}(z)\|_E = O\left(n^{2(\bar{\mu}_\kappa + \bar{\mu}_{\kappa+1} - 2)} \left|\frac{z_\kappa}{z_{\kappa+1}}\right|^{2n}\right). \tag{4.3a}$$

The bivector part of the denominator converges uniformly to zero in E*, with the rate of convergence governed by*

$$\|A^{l,m}(z)\|_E = O\left(n^{\bar{\mu}_\kappa + \bar{\mu}_{\kappa+1} - 2} \left|\frac{z_\kappa}{z_{\kappa+1}}\right|^{n}\right). \tag{4.3b}$$

If (4.2) *does not hold then* (4.3*b*) *is still valid while* (4.3*a*) *is replaced by*

$$\|R_\kappa(z) - \sigma^{l,m}(z)\|_E = O\left(n^{\bar{\mu}_\kappa + \bar{\mu}_{\kappa+1} - 2} \left|\frac{z_\kappa}{z_{\kappa+1}}\right|^{n}\right). \tag{4.4}$$

For functions of the form considered in Section 3 *statements similar to* (4.3*b*) *and* (4.4) *hold.*

Proof. We follow the proof of theorem (3.1) and seek to apply the conditions (3.16) to the unknowns in (3.10). Observe that $F_{n,r}(z_k)$ of (3.19) may be expressed as

$$F_{n,r}(z_k) = \frac{(z_k)^{n+1}}{2\pi i} \oint_{|v|=\rho_M} \frac{\mathbf{g}_\kappa(v)\, q^{[l/m]}(v)}{v^{n+1}(v-z_k)^{r+1}}\, dv \\ -(z_k)^{n+1} \sum_{k'=\kappa+1}^{M} \operatorname{Res}\left[\frac{h^{l,m}(v)}{v^{n+1}(v-z_k)^{r+1}} : v = z_{k'}\right], \tag{4.5}$$

where the contour of (3.19) has been expanded to ρ_M satisfying $|z_M| < \rho_M$.

The integral in (4.5) may be shown to vanish for $n \geqslant 2m + \mu_{M+1} - r$. Using (3.5), (3.10) , and the definition of $h^{l,m}(v)$ we may write

$$\frac{h^{l,m}(v)}{(v-z_k)^{r+1}} = \mathbf{f}(v) \left[q_{n,m} T^{k,r}(v) + \sum_{k_1=1}^{\kappa} \sum_{s_1=0}^{\mu_{k_1}-1} q_n^{(s_1)}(z_{k_1})\, T^{k,r}_{k_1,s_1}(v) \right] \tag{4.6}$$

in which $T^{k,r}(v)$ and $T^{k,r}_{k_1,s_1}(v)$ are scalar polynomials, whose coefficients are independent of n, each of maximum degree $(2m-r-1)$. Then, considering the contribution of the first term in (4.6) to (4.5), we apply Leibnitz's theorem to write

$$\operatorname{Res}\left[\frac{\mathbf{f}(v)\, T^{k,r}(v)}{v^{n+1}} : v = z_{k'}\right] = \sum_{k'=\kappa+1}^{M} \sum_{s'=0}^{\mu_{k'}-1} \mathbf{v}_{k',s'} \operatorname{Res}\left[\frac{T^{k,r}(v)}{v^{n+1}(v-z_{k'})^{s'+1}} : v = z_{k'}\right] \\ = \frac{1}{(z_{k'})^{n+1}} \sum_{k'=\kappa+1}^{M} \sum_{s'=0}^{\mu_{k'}-1} n^{s'} \mathbf{v}_{k',s'} \alpha^{n,k,r}_{k',s'},$$

where

$$\alpha^{n,k,r}_{k',s'} := \frac{-1}{n^{s'}} \sum_{l=0}^{s'} \frac{(-1)^l\,(n+1)_l}{(z_{k'})^l\, l!(s'-l)!} \left[\frac{d^{(s'-l)}}{dv^{(s'-l)}} T^{k,r}(v)\right]_{v=z_{k'}}.$$

For fixed k, r, k', s' the $\alpha^{n,k,r}_{k',s'}$ form a sequence of complex numbers which is bounded as $n \to \infty$. Treating the remaining terms of (4.6) in similar fashion leads to

$$F_{n,r}(z_k) = \sum_{k',s'} \left[\frac{z_k}{z_{k'}}\right]^{n+1} n^{s'} \mathbf{v}_{k',s'} \left[q_{n,m} \alpha^{n,k,r}_{k',s'} + \sum_{k_1,s_1} q_n^{(s')}(z_{k_1})\, \alpha^{n,k,r}_{k_1,s_1;k',s'} \right] \tag{4.7}$$

in which, for fixed k, r, k_1, s_1, k', s' the $\alpha^{n,k,r}_{k_1,s_1;k',s'}$ form another sequence of complex numbers bounded as $n \to \infty$. For brevity, we adopt the convention that unprimed k-dependent indices take the values $1, 2, \ldots, \kappa$, while if primed they run from $\kappa+1$ to M.

Hence, using (4.7), (3.18), i.e., the left hand side of (3.17), may be expressed as

$$s! \sum_{r=0}^{s} \sum_{k', s'} \binom{n+1}{s-r} (z_k)^{r-s} \left[\frac{z_k}{z_{k'}}\right]^{n+1} n^{s'} \mathbf{v}_{k', s'} \left[q_{n, m} \alpha_{k', s'}^{n, k, r} + \sum_{k_1, s_1} q_n^{(s_1)}(z_{k_1})\, \alpha_{k_1, s_1; k', s'}^{n, k, r} \right]$$

$$= n^{s+\bar{\mu}_{\kappa+1}-1} \left[\frac{z_k}{z_{\kappa+1}}\right]^n \sum_{k', s'} \mathbf{v}_{k', s'} \left[q_{n, m} \beta_{k', s'}^{n, k} + \sum_{k_1, s_1} q_n^{(s_1)}(z_{k_1})\, \beta_{k_1, s_1; k', s'}^{n, k} \right], \tag{4.8}$$

where we have introduced the complex numbers

$$\beta_{k', s'}^{n, k} := \frac{s!}{n^{s+\bar{\mu}_{\kappa+1}-1-s'} z_{k'}} \sum_{r=0}^{s} \binom{n+1}{s-r} (z_k)^{r+1-s} \alpha_{k', s'}^{n, k, r} \left[\frac{z_{\kappa+1}}{z_{k'}}\right]^n$$

which remain bounded as $n \to \infty$ for fixed k, k', s'. Similar definitions apply for the bounded sequences $\beta_{k, s; k', s'}^{n, k}$.

We define new vector sequences by

$$\mathbf{V}^{(n, k)} := \sum_{k', s'} \mathbf{v}_{k', s'} \beta_{k', s'}^{n, k} \tag{4.9a}$$

and

$$\mathbf{V}_{k_1, s_1}^{(n, k)} := \sum_{k', s'} \mathbf{v}_{k', s'} \beta_{k_1, s_1; k', s'}^{n, k} \tag{4.9b}$$

which are $O(1)$ as $n \to \infty$ keeping the other indices fixed.

The conditions originating from (3.16) now may be stated as

$$\mathbf{u}_k q_n^{(s)}(z_k) + \sum_{r=0}^{s-1} \binom{s}{r} \mathbf{u}_{k, s-r} q_n^{(r)}(z_k)$$

$$= n^{s+\bar{\mu}_{\kappa+1}-1} \left[\frac{z_k}{z_{\kappa+1}}\right]^n \left[q_{n, m} \mathbf{V}^{(n, k)} + \sum_{k_1, s_1} q_n^{(s_1)}(z_{k_1})\, \mathbf{V}_{k_1, s_1}^{(n, k)} \right] \tag{4.10}$$

with

$$\mathbf{u}_k := \mathbf{g}_\kappa(z_k) \qquad \text{and} \qquad \mathbf{u}_{k, s} := \mathbf{g}_\kappa^{(s)}(z_k) \tag{4.11}$$

for $k = 1, 2 \cdots \kappa$ and $s = 1, 2, \ldots, \mu_k - 1$. Again we emphasise that $\mathbf{u}_k$, from (3.4), is invertible. We note also that

$$\mathbf{u}_{k, s} = s! \sum_{r=0}^{s} v_{k, s-r} \mathbf{v}_{k, \mu_k - r - 1} \qquad \text{with} \quad \mathbf{u}_k = \mathbf{u}_{k, 0},$$

where $v_{k, s} := R_\kappa^{(\mu_k + s)}(z_k)/(\mu_k + s)!$ for $k = 1, 2, \ldots, \kappa$ and $s = 0, 1, \ldots, \mu_k - 1$. If we denote the vector space spanned by the $\mathbf{v}_{k, s}$, for $s = 0, 1 \cdots \mu_k - 1$, by

$\mathscr{V}_k$ then $\mathbf{u}_k, \mathbf{u}_{k,r} \in \mathscr{V}_k$, $r = 0, 1 \cdots \mu_k - 1$ and $\mathscr{V}_{k_1} \perp \mathscr{V}_{k_2}$, $k_1 \neq k_2$ from (4.2). Also, $\mathbf{V}^{(n,k)}$ and $\mathbf{V}^{(n,k)}_{k_1,s_1}$ are orthogonal to $\mathscr{V}_{k_2}$ for $k_2 = 1, 2, \ldots, \kappa$.

We solve (4.10) iteratively. Defining the Clifford element

$$W^{k}_{n} := n^{\bar{\mu}_{\kappa+1}-1}\left[\frac{z_k}{z_{\kappa+1}}\right]^n \mathbf{u}_k{}^{-1}\left[\mathbf{V}^{(n,k)} + \sum_{k_1,s_1}\left\{\frac{q_n^{(s_1)}(z_{k_1})}{q_{n,m}}\right\}\mathbf{V}^{(n,k)}_{k_1,s_1}\right] \tag{4.12}$$

Eq. (4.10) may be rewritten as

$$\hat{q}_n^{(s)}(z_k) := \left[\frac{q_n^{(s)}(z_k)}{q_{n,m}}\right] = n^s W^k_n - \sum_{r=0}^{s-1}\binom{s}{r}\mathbf{u}_k{}^{-1}\mathbf{u}_{k,s-r}\hat{q}_n^{(r)}(z_k). \tag{4.13}$$

By repeated substitution we may remove the unknowns in the summation on the right-hand side of (4.13) to obtain

$$\hat{q}_n^{(s)}(z_k) = n^s Q^{(k,s)}\left(\frac{1}{n}\right) W^k_n, \qquad k = 1, 2, \ldots, \kappa \quad s = 0, 1, \ldots, \mu_k - 1, \tag{4.14}$$

where the $Q^{(k,s)}(1/n)$ are polynomials in $1/n$ over $Cl(\mathbb{C}^d)$, each of maximum degree s, and which satisfy the recurrence relations

$$Q^{(k,s+1)}\left(\frac{1}{n}\right) = 1 - \sum_{r=0}^{s}\binom{s+1}{r}\mathbf{u}_k{}^{-1}\mathbf{u}_{k,s+1-r}Q^{(k,r)}\left(\frac{1}{n}\right)\left[\frac{1}{n}\right]^{s+1-r} \tag{4.15}$$

with

$$Q^{(k,0)}\left(\frac{1}{n}\right) := 1. \tag{4.16}$$

We may prove by induction that, for the positive numbers γ_k $(k = 1, 2, \ldots, \kappa)$ given by

$$\gamma_k := \max_{\substack{0 \leqslant r \leqslant s \\ 0 \leqslant s < \mu_k}}\left\{\binom{s+1}{r} K_d\,|\mathbf{u}_k{}^{-1}\mathbf{u}_{k,s+1-r}|\right\}$$

then

$$\left|Q^{(k,s)}\left(\frac{1}{n}\right)\right| \leqslant (1+\gamma_k)^s, \qquad s = 0, 1, \ldots, \mu_k - 1$$

using the spinor norm (2.6).

It is readily seen that, apart from scalar factors dependent on binomial coefficients, the Clifford coefficients of $Q^{(k,r)}(\frac{1}{n})$ are sums of products of **even** numbers of vectors in $\mathbb{C}^d$ of the form

$$\mathbf{u}_k{}^{-1}\mathbf{u}_{k,i_1}\mathbf{u}_k{}^{-1}\mathbf{u}_{k,i_2}\cdots\mathbf{u}_k{}^{-1}\mathbf{u}_{k,i_t}, \qquad t \leqslant r. \tag{4.17}$$

Using (4.12), Eq. (4.14) may be rewritten as

$$\hat{q}_n^{(s)}(z_k) = a_n^{k,s} + \sum_{k_1, s_1} b_n^{k,s;k_1,s_1} \hat{q}_n^{(s_1)}(z_{k_1}), \tag{4.18}$$

where

$$a_n^{k,s} := n^{s+\bar{\mu}_{\kappa+1}-1} \left[\frac{z_k}{z_{\kappa+1}}\right]^n Q^{(k,s)}\left(\frac{1}{n}\right) \mathbf{u}_k{}^{-1} \mathbf{V}^{n,k} \tag{4.19}$$

and

$$b_n^{k,s;k_1,s_1} := n^{s+\bar{\mu}_{\kappa+1}-1} \left[\frac{z_k}{z_{\kappa+1}}\right]^n Q^{(k,s)}\left(\frac{1}{n}\right) \mathbf{u}_k{}^{-1} \mathbf{V}^{n,k}_{k_1,s_1}. \tag{4.20}$$

We note that both $a_n^{k,s}$ and $b_n^{k,s;k_1,s_1}$ are of the same order

$$|a_n^{k,s}|, |b_n^{k,s;k_1,s_1}| = O\left(n^{s+\bar{\mu}_{\kappa+1}-1} \left|\frac{z_k}{z_{\kappa+1}}\right|^n\right) \tag{4.21}$$

thus guaranteeing, for sufficiently large n, the convergence of the following infinite series representing the solution to (4.18)

$$\hat{q}_n^{(s)}(z_k) = \sum_{t=0}^{\infty} C_{n,t}^{k,s}, \tag{4.22}$$

where

$$C_{n,0}^{k,s} := a_n^{k,s} \qquad \text{and}$$

$$C_{n,t}^{k,s} := \sum_{k_1,s_1} \cdots \sum_{k_t,s_t} b_n^{k,s;k_1,s_1} b_n^{k_1,s_1;k_2,s_2} \cdots b_n^{k_{t-1},s_{t-1};k_t,s_t} a_n^{k_t,s_t}. \tag{4.23}$$

However, our interest is in the scalar part of $[q_n^{(s)}(z_k)/q_{n,m}]$. Therefore, we consider the partial sum

$$S_{n,L}^{k,s} := \sum_{t=0}^{L} C_{n,t}^{k,s} \tag{4.24}$$

whose scalar part is denoted by

$$\langle S_{n,L}^{k,s} \rangle_0 = \sum_{t=0}^{L} \langle C_{n,t}^{k,s} \rangle_0. \tag{4.25}$$

From (4.19) we observe that

$$\langle C_{n,0}^{k,s} \rangle_0 = n^{s+\bar{\mu}_{\kappa+1}-1} \left[\frac{z_k}{z_{\kappa+1}} \right]^n \left\langle Q^{(k,s)}\left(\frac{1}{n}\right) \mathbf{u}_k{}^{-1} \mathbf{V}^{(n,k)} \right\rangle_0. \tag{4.26}$$

The factor $\langle \cdots \rangle_0$ on the right-hand side may be expanded as a sum of terms, a typical example of which is

$$\langle \mathbf{u}_k{}^{-1} \mathbf{u}_{k_1,i_1} \mathbf{u}_k{}^{-1} \cdots \mathbf{u}_k{}^{-1} \mathbf{u}_{k_1,i_t} \mathbf{u}_k{}^{-1} \mathbf{V}^{(n,k)} \rangle_0 \tag{4.27}$$

using (4.17). However, the scalar part of a product of an odd number of vectors is zero, while for an even number of vectors $\mathbf{v}_i \in \mathbb{C}^d$, $i = 1, 2, ..., 2t$

$$\langle \mathbf{v}_1 \mathbf{v}_2 \cdots \mathbf{v}_{2t} \rangle_0 = \sum_{\mathbf{i}} \sigma(\mathbf{i})(\mathbf{v}_{i_1} \cdot \mathbf{v}_{i_2})(\mathbf{v}_{i_3} \cdot \mathbf{v}_{i_4}) \cdots (\mathbf{v}_{i_{2t-1}} \cdot \mathbf{v}_{i_{2t}}), \tag{4.28}$$

where the summation is over all permutations $\mathbf{i} := (i_1, i_2, ..., i_{2t})$ of the integers $1, 2, ..., 2t$ and $\sigma(\mathbf{i})$ denotes the corresponding parity of the permutation [15]. Therefore, since, from the comments made in the paragraph before Eq. (4.12)

$$\mathbf{u}_k \cdot \mathbf{V}^{(n,k)} = \mathbf{u}_{k,i} \cdot \mathbf{V}^{(n,k)} = 0, \qquad i = 1, ..., \mu_k - 1$$

we conclude that

$$\langle C_{n,0}^{k,s} \rangle_0 = 0. \tag{4.29}$$

We now consider (cf. (4.23))

$$\begin{aligned} C_{n,t}^{k,s} = \sum_{k_1,s_1} \cdots \sum_{k_t,s_t} & n^{(t+1)(\bar{\mu}_{\kappa+1}-1)+(s+s_1+\cdots+s_t)} \left[\frac{z_k z_{k_1} \cdots z_{k_t}}{(z_{\kappa+1})^{t+1}} \right]^n \\ & \times Q^{(k,s)}\left(\frac{1}{n}\right) \mathbf{u}_k{}^{-1} \mathbf{V}_{k_1,s_1}^{(n,k)} Q^{(k_1,s_1)}\left(\frac{1}{n}\right) \mathbf{u}_{k_1}{}^{-1} \mathbf{V}_{k_2,s_2}^{(n,k_1)} \cdots \\ & \times \mathbf{V}_{k_t,s_t}^{(n,k_{t-1})} Q^{(k_t,s_t)}\left(\frac{1}{n}\right) \mathbf{u}_{k_t}{}^{-1} \mathbf{V}^{(n,k_t)}. \end{aligned} \tag{4.30}$$

When expanded in terms of a sum of products of vectors, each product involves $(t+1)$ vectors from $\bigoplus \mathscr{V}_{k'}$, $(k' = \kappa+1, ..., M)$, each of which is orthogonal to all the other vectors originating from $\mathscr{V}_k$, $(k = 1, 2, ..., \kappa)$. Hence, for a non-zero contribution to the scalar part of this product t must be odd. In addition, the expansion of the factor $Q^{(k,s)}(1/n)\, \mathbf{u}_k{}^{-1}$ contains an odd number of vectors from $\mathscr{V}_k$ in each product. From (4.28) one of these vectors must be contracted (i.e., a scalar product composed) with a vector from $\mathscr{V}_{k_i}$, $(i = 1, 2, ..., t)$. For a non-zero contribution, this can only happen if $k = k_i$ for at least one i between 1 and t.

We also point out that, since $Q^{(k,s)}(1/n)$ is bounded for fixed k, s, and since $\mathbf{V}^{(n,k)}_{k',s'}[\mathbf{V}^{(n,k)}]$ are bounded vectors for fixed k, k', s', which integers may assume a finite number (independent of n) of values, there is a constant β independent of k, s, k', s', n such that

$$\left|Q^{(k,s)}\left(\frac{1}{n}\right)\mathbf{u}_k{}^{-1}\mathbf{V}^{(n,k)}_{k',s'}\right| \leqslant \beta \quad \text{and} \quad \left|Q^{(k,s)}\left(\frac{1}{n}\right)\mathbf{u}_k{}^{-1}\mathbf{V}^{(n,k)}\right| \leqslant \beta.$$

Therefore,

$$|\langle C^{k,s}_{n,t}\rangle_0| \leqslant \beta^2 K_d n^{s+\mu_k+2\bar{\mu}_{\kappa+1}-3}\left|\frac{z_k}{z_{\kappa+1}}\right|^{2n}\left\{mK_d\beta n^{\bar{\mu}_\kappa+\bar{\mu}_{\kappa+1}-2}\left|\frac{z_\kappa}{z_{\kappa+1}}\right|^n\right\}^{t-1}. \tag{4.31}$$

Hence, for n sufficiently large that $|\{mK_d\cdots\}| < 1/\sqrt{2}$, we obtain

$$|\langle S^{k,s}_{n,L}\rangle_0| \leqslant 2\beta^2 K_d n^{s+\mu_k+2\bar{\mu}_{\kappa+1}-3}\left|\frac{z_k}{z_{\kappa+1}}\right|^{2n}. \tag{4.32}$$

We conclude that

$$|\langle \hat{q}^{(s)}_n(z_k)\rangle_0| \leqslant 2\beta^2 K_d n^{s+\mu_k+2\bar{\mu}_{\kappa+1}-3}\left|\frac{z_k}{z_{\kappa+1}}\right|^{2n}. \tag{4.33}$$

Following (3.29) we finally obtain, for $z \in E$, a compact subset of $\mathbb{C}$,

$$\begin{aligned} |R_\kappa(z) - \langle \hat{q}^{[l/m]}(z)\rangle_0| &\leqslant \sum_{k=1}^{\kappa}\sum_{s=0}^{\mu_k-1}|B_{k,s}(z)|\, 2\beta^2 K_d n^{s+\mu_k+2\bar{\mu}_{\kappa+1}-3}\left|\frac{z_k}{z_{\kappa+1}}\right|^{2n} \\ &\leqslant K_\perp n^{2(\bar{\mu}_\kappa+\bar{\mu}_{\kappa+1}-2)}\left|\frac{z_\kappa}{z_{\kappa+1}}\right|^{2n} \end{aligned} \tag{4.34}$$

for sufficiently large n and an appropriate constant $K_\perp$ independent of n, thus proving (4.3a).

For the bivector part of $\hat{q}^{[l/m]}(z)$ we use (4.18), (4.21), and (4.22) to write

$$|\langle \hat{q}^{(s)}_n(z_k)\rangle_2| = O\left(n^{s+\bar{\mu}_{\kappa+1}-1}\left|\frac{z_k}{z_{\kappa+1}}\right|^n\right). \tag{4.35}$$

Therefore, with the help of (3.10) we obtain

$$|\langle \hat{q}^{[l/m]}(z)\rangle_2| = O\left(n^{\bar{\mu}_\kappa+\bar{\mu}_{\kappa+1}-2}\left|\frac{z_\kappa}{z_{\kappa+1}}\right|^n\right) \tag{4.36}$$

for $z \in E$. The rate of convergence in (4.35) and hence (4.36) cannot be improved, in general, since there is a non-zero contribution from the bivector $\mathbf{u}_k \wedge \mathbf{V}^{(n,k)}$ which vanishes if and only if the two vectors are **parallel**. This is not the case in general—indeed, for real non-null vectors this is impossible, since the two vectors are orthogonal. Statement (4.3*b*) now follows.

In the situation where the vectors $\mathbf{v}_{k,s}$ of (4.1) do not satisfy (4.2) it is clear from (4.18), (4.21), (4.22), and (3.28) that

$$|\langle \hat{q}_n^{(s)}(z_k)\rangle_0|, |\langle \hat{q}_n^{(s)}(z_k)\rangle_2| = O\left(n^{s+\bar{\mu}_{\kappa+1}-1}\left|\frac{z_k}{z_{\kappa+1}}\right|^n\right). \tag{4.37}$$

Indeed, from (3.27) and (3.28), this is true for functions of the form depicted in Section 3, The remaining results of the theorem then follow. ▮

5. POLE APPROXIMATION

In this section we consider various approximations to the poles of vector-valued functions $\mathbf{f}(z)$, of the forms described in Sections 3 and 4. The poles of the $[l/m]$ vector Padé approximant to $\mathbf{f}(z)$ are given by the zeroes of $Q^{l,m}(z)$ defined by (2.16). However, this polynomial is of degree $2m$, thus yielding twice as many poles as required. Since the bivector part of the denominator converges to zero as $l \to \infty$, an alternative approach is to approximate $R_\kappa(z)$ with the scalar part of $\hat{q}^{[l/m]}(z)$, *viz.* $\sigma^{l,m}(z)$. This polynomial has the correct number of zeroes which we denote by $z_{k,j}^{(n)}$ ($j = 1, 2, ..., \mu_k$, $k = 1, 2, ..., \kappa$). Furthermore, we may follow Sidi [28], and construct more accurate approximations to multipoles z_k for $k \in \{1, 2, ..., \kappa\}$, by taking averages

$$\hat{z}_k^{(n)} := \frac{1}{\mu_k} \sum_{j=1}^{\mu_k} z_{k,j}^{(n)} \tag{5.1}$$

or by defining the polynomial

$$\bar{\sigma}_k^{l,m}(z) := \frac{d^{(\mu_k-1)}}{dz^{(\mu_k-1)}} \sigma^{l,m}(z) \tag{5.2}$$

of degree $(m - \mu_k + 1)$, which may be shown to have a simple zero, $\bar{z}_k^{(n)}$, near z_k for sufficiently large n.

Whichever approximation we employ the rate of convergence determined for vector-valued functions obeying the orthogonality condition (4.2) is twice that for other functions.

Theorem 5.1 refers to $\sigma^{l,m}(z)$ and related approximations, while Theorem 5.2 treats $Q^{l,m}(z)$ in a similar manner.

THEOREM 5.1. *For sufficiently large n, the scalar part of the monic denominator polynomial,* $\sigma^{l,m}(z)$, *of the* $[l/m]$ *vector Padé approximant to a vector-valued function* $\mathbf{f}(z)$ *of Theorem* 3.1 *has, for each value of* $k \in \{1, 2, \ldots, \kappa\}$, μ_k *zeroes* $z_{k,j}^{(n)}$ $(j = 1, 2, \ldots, \mu_k)$ *in the neighbourhood of* z_k, *satisfying*

$$\epsilon_{k,j}^{(n)} := z_k - z_{k,j}^{(n)} = O\left(n^{\tau(\bar{\mu}_{\kappa+1}-1)/\mu_k} \left|\frac{z_k}{z_{\kappa+1}}\right|^{\tau n/\mu_k}\right) \tag{5.3}$$

and

$$\hat{\epsilon}_k^{(n)} := z_k - \hat{z}_k^{(n)} = O\left(n^{\tau(\mu_k+\bar{\mu}_{\kappa+1}-2)} \left|\frac{z_k}{z_{\kappa+1}}\right|^{\tau n}\right), \tag{5.4}$$

where $\tau = 1$.

Furthermore, for sufficiently large n, the polynomial $\bar{\sigma}_k^{l,m}(z)$ *has a simple zero,* $\bar{z}_k^{(n)}$, *near* z_k, *satisfying*

$$\bar{\epsilon}_k^{(n)} := z_k - \bar{z}_k^{(n)} = O\left(n^{\tau(\mu_k+\bar{\mu}_{\kappa+1}-2)} \left|\frac{z_k}{z_{\kappa+1}}\right|^{\tau n}\right). \tag{5.5}$$

If the function $\mathbf{f}(z)$ *satisfies the conditions of Theorem* 4.1, *then* $\tau = 2$ *in* (5.3), (5.4), (5.5).

Proof. Since $\sigma^{l,m}(z)$ tends to $R_\kappa(z)$ uniformly in compact subsets of $\mathbb{C}$, each of the m zeroes, $z_{k,j}^{(n)}$ $(k = 1, 2, \ldots, \kappa; j = 1, 2, \ldots, \mu_k)$, of the former polynomial must tend to a zero of $R_\kappa(z)$. Hence, for each $k \in \{1, 2, \ldots, \kappa\}$, and for sufficiently large n, all $z_{k,j}^{(n)}$, for $j = 1, 2, \ldots, \mu_k$ must lie close to z_k. We set

$$\sigma^{l,m}(z) = a_{k,n}(z) \prod_{j=1}^{\mu_k} (z - z_{k,j}^{(n)}), \tag{5.6}$$

where $a_{k,n}(z)$ are monic polynomials over $\mathbb{C}$ of maximum degree $(m - \mu_k)$. It is clear that there must exist positive numbers δ, n_δ, such that, for all $n > n_\delta$

$$|a_{k,n}(z_k)| \geqslant \delta, \qquad k = 1, 2, \ldots, \kappa. \tag{5.7}$$

Noting also that

$$\langle \hat{q}_n^{(s)}(z_k) \rangle_0 = O(1), \qquad s = \mu_k, \mu_k + 1, \ldots, m \tag{5.8}$$

we may use Taylor's theorem to write

$$0=\sigma^{l,m}(z_{k,j}^{(n)})=\sum_{s=0}^{\mu_k-1}\langle\hat{q}_n^{(s)}(z_k)\rangle_0\,[-\epsilon_{k,j}^{(n)}]^s+[-\epsilon_{k,j}^{(n)}]^{\mu_k}\,O(1). \tag{5.9}$$

Statement (5.3) then follows using (5.7) with (3.27) and (3.28).

To prove (5.4) we observe that

$$\langle\hat{q}_n^{(s)}(z_k)\rangle_0=\sum_{i=0}^{s}\frac{s!}{(s-i)!}\,a_{k,n}^{(s-i)}(z_k)\,p_{k,\mu_k-i}(\boldsymbol{\varepsilon}_k^{(n)}) \qquad s=0,1,\ldots,m, \tag{5.10}$$

where the $p_{k,i}(\mathbf{x})$ are symmetric polynomials defined by

$$\prod_{i=1}^{\mu_k}(1+zx_i)=\sum_{i=0}^{\mu_k}z^ip_{k,i}(\mathbf{x}), \qquad \mathbf{x}:=(x_1,x_2\cdots,x_{\mu_k})\in\mathbb{C}^{\mu_k} \tag{5.11}$$

and $\boldsymbol{\varepsilon}_k^{(n)}:=(\epsilon_{k,1}^{(n)},\epsilon_{k,2}^{(n)},\ldots,\epsilon_{k,\mu_k}^{(n)})$. We consider functions of the form discussed in theorem 3.1. It is readily shown using proof by induction, that (3.27), (5.7), and (5.8) imply

$$p_{k,\mu_k-s}(\boldsymbol{\varepsilon}_k^{(n)})=O\left(n^{s+\bar{\mu}_{\kappa+1}-1}\left|\frac{z_k}{z_{\kappa+1}}\right|^n\right), \qquad s=0,1,\ldots,\mu_k-1. \tag{5.12}$$

In particular,

$$\sum_{j=1}^{\mu_k}\epsilon_{k,j}^{(n)}=p_{k,1}(\boldsymbol{\varepsilon}_k^{(n)})=O\left(n^{\mu_k+\bar{\mu}_{\kappa+1}-2}\left|\frac{z_k}{z_{\kappa+1}}\right|^n\right). \tag{5.13}$$

The result (5.4) with $\tau=1$ then follows.

Turning to the last part of the theorem, we note that (5.8) allows us to state

$$0=\bar{\sigma}_k^{l,m}(\bar{z}_k^{(n)})=\langle\hat{q}_n^{(\mu_k-1)}(z_k)\rangle_0-\bar{\epsilon}_k^{(n)}\langle\hat{q}_n^{(\mu_k)}(z_k)\rangle_0+O(|\bar{\epsilon}_k^{(n)}|^2). \tag{5.14}$$

Using (5.7) with (3.27) and (3.28) we obtain

$$\bar{\epsilon}_k^{(n)}=O\left(n^{\mu_k+\bar{\mu}_{\kappa+1}-2}\left|\frac{z_k}{z_{\kappa+1}}\right|^n\right). \tag{5.15}$$

In fact, we may demonstrate that $\hat{\epsilon}_k^{(n)}$ and $\bar{\epsilon}_k^{(n)}$ agree to leading order.

The remaining parts of the theorem follow using (4.33) instead of (3.27) and (3.28). ▮

We now consider the scalar polynomial of degree $2m$, $Q^{l,m}(z)$ of (2.16) in which $q^{[l/m]}(z)$ is monic. From Corollary 3.3 we observe that, for sufficiently large n, there are $2\mu_k$ zeroes of $Q^{l,m}(z)$ close to z_k for each $k\in\{1,2,\ldots,\kappa\}$. We label these zeroes $z'^{(n)}_{k,j}$ $(j=1,2,\ldots,2\mu_k)$. For a given k

the following theorem considers not only the behaviour of these zeroes, but also that of their average, $\hat{z}'^{(n)}_k$. In addition we form the polynomial

$$\bar{Q}^{l,m}_k(z) := \frac{d^{(2\mu_k - 1)}}{dz^{(2\mu_k - 1)}} Q^{l,m}(z) \tag{5.16}$$

which has a simple zero near z_k for large enough n.

THEOREM 5.2. *For sufficiently large n, the* $[l/m]$ *vector Padé approximant to a vector-valued function* $\mathbf{f}(z)$ *of Theorem* 3.1 *has, for each value of* $k \in \{1, 2, \dots, \kappa\}$, $2\mu_k$ *poles* $z'^{(n)}_{k,j}$, $j = 1, 2, \dots, 2\mu_k$ *in the neighbourhood of* z_k, *satisfying*

$$\epsilon'^{(n)}_{k,j} := z_k - z'^{(n)}_{k,j} = O\left(n^{(\bar{\mu}_{\kappa+1}-1)/\mu_k} \left|\frac{z_k}{z_{\kappa+1}}\right|^{n/\mu_k}\right) \tag{5.17}$$

and

$$\hat{\epsilon}'^{(n)}_k := z_k - \hat{z}'^{(n)}_k = O\left(n^{\mu_k + \bar{\mu}_{\kappa+1} - 2} \left|\frac{z_k}{z_{\kappa+1}}\right|^{n}\right). \tag{5.18}$$

Furthermore, for sufficiently large n, the polynomial $\bar{Q}^{l,m}_k(z)$ *has a simple zero,* $\bar{z}'^{(n)}_k$, *near* z_k, *satisfying*

$$\bar{\epsilon}'^{(n)}_k := z_k - \bar{z}'^{(n)}_k = O\left(n^{\mu_k + \bar{\mu}_{\kappa+1} - 2} \left|\frac{z_k}{z_{\kappa+1}}\right|^{n}\right). \tag{5.19}$$

Proof. The arguments are similar to those used to derive the previous theorem. Hence, we simply present some of the differences here. From the definition of $Q^{l,m}(z)$, (3.27), (3.28), and (5.8), it may be shown that

$$\frac{d^s}{dz^s} Q^{l,m}(z)\Big|_{z_k} = \begin{cases} O\left(n^{s+2\bar{\mu}_{\kappa+1}-2} \left|\dfrac{z_k}{z_{\kappa+1}}\right|^{2n}\right), & s = 0, 1, \dots, \mu_k - 1 \\ O\left(n^{s-\mu_k+\bar{\mu}_{\kappa+1}-1} \left|\dfrac{z_k}{z_{\kappa+1}}\right|^{n}\right), & s = \mu_k, \mu_k + 1, \dots, 2\mu_k - 1. \end{cases} \tag{5.20}$$

A statement similar to (5.7) may be made, while derivatives of order higher than the $2\mu_k^{th}$ may be shown to be $O(1)$; cf. (5.8). Corresponding to (5.12) we have

$$p_{k, 2\mu_k - s}(\varepsilon'^{(n)}_k) = \begin{cases} O\left(n^{s+2\bar{\mu}_{\kappa+1}-2} \left|\dfrac{z_k}{z_{\kappa+1}}\right|^{2n}\right), & s = 0, 1, \dots, \mu_k - 1 \\ O\left(n^{s-\mu_k+\bar{\mu}_{\kappa+1}-1} \left|\dfrac{z_k}{z_{\kappa+1}}\right|^{n}\right), & s = \mu_k, \mu_k + 1, \dots, 2\mu_k - 1. \end{cases} \tag{5.21}$$

On setting $s = 2\mu_k - 1$, (5.18) is confirmed. ▮

Even if $\mathbf{f}(z)$ satisfies the conditions of Theorem 4.1, in general the rate of convergence is not improved. In this respect note the convergence behaviour of the bivector given by (4.35).

We observe that the above results are of a nature similar to those derived in [28], using a different definition of approximation.

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